A Fourier transform-based method for convertible bonds in a jump diffusion setting with stochastic interest rates

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Abstract

The paper proposes a fast Fourier transform (FFT) pricing algorithm for convertible bonds in a framework which comprises firm value, evolving as an exponential jump diffusion, and correlated stochastic interest rates movements. This is a novel numerical technique for the convertible bonds literature and aims at fixing dimensionality and convergence limitations previously reported for popular grid-based, lattice and Monte Carlo methods. We define the firm’s optimal call policy and investigate its impact on the computed convertible bond prices. We illustrate the performance of the numerical scheme and highlight the effects originated by the inclusion of jumps.
I Introduction

The aim of this paper is to introduce a Fourier transform approach for the pricing of convertible bonds (CBs) under a jump diffusion market model with correlated stochastic interest rates. In contrast with the previous literature, the proposed numerical pricing technique can accommodate a number of risk factors and contract-design features, and is shown to be efficient and accurate.

CBs are hybrid instruments which represent a pricing challenge because of their complex design. In first place, they depend on variables related to the underlying firm value (or stock), the fixed income part, which includes both interest rates and default risk, and the interaction between these components. Further, CBs usually carry call options giving the issuer the right to demand premature redemption in exchange for the current call price. Put option features, which allow the investor to force the issuing firm to prematurely repurchase the CB for a pre-specified price, are also sometimes met.

The early-exercise features that CBs present imply that the pricing problem of these contracts shares strong analogies with the one of American/Bermudan vanilla options. Closed-form solutions for the price of the CB in a Black-Scholes-Merton economy have been obtained by Ingersoll (1977a) for the case of non-callable/callable products; however, the introduction in the valuation model of a more realistic specification including, for instance, discretely payable coupons, dividends on the underlying stock, soft call provisions (which preclude the issuer from calling the CB until the firm value rises above a specified level), and a call notice period prevent the derivation of explicit pricing formulae. For these reasons, various numerical techniques have been employed in order to evaluate CBs. The literature distinguishes mainly between three types of approach: (i) numerical schemes for partial differential equations/inequalities (PDE/Is) (see, for instance, Brennan and Schwartz (1977), (1980), Carayannopoulos (1996), Tsiveriotis and Fernandes (1998), Zvan et al. (1998), (2001), Takahashi et al. (2001), Barone-Adesi et al. (2003) and Bermúdez and Webber (2004)), (ii) lattice methods (see, for example, Goldman
Sachs (1994), Ho and Pfeffer (1996), Takahashi et al. (2001) and Davis and Lischka (2002)) and (iii) Monte Carlo simulation (see Lvov et al. (2004), for an approach based on the joint simulation-regression technique by Longstaff and Schwartz (2001), and Ammann et al. (2008), for an approach based, instead, on the optimization method by García (2003)).

Contributions using the PDE/I approach rely on the finite difference (Brennan and Schwartz (1977), (1980)) and finite volume (Zvan et al. (2001)) schemes; a more recent development is the so-called joint characteristics-finite elements method proposed by Barone-Adesi et al. (2003), which aims at overcoming previously reported challenges originated by complex boundary conditions, the existence of spurious oscillations due to convection dominance, and the slow convergence. On the contrary, the popularity of lattices is frequently attributed to their intuitiveness and simplicity; lattice methods, though, suffer of increasing number of spatial nodes at each time step, especially for long maturities. This issue becomes even more noticeable in the case in which stochastic interest rates are included, as this requires the generation of a 2-D lattice. This problem could be handled by using a small number of large time steps, which would inevitably affect, though, the accuracy of the scheme. Further, Geske and Shastri (1985) demonstrate that lattices tend to lose efficiency when dealing with discrete payments and early-exercise options.

One significant problem with the traditional PDE/I and lattice methods is the so-called curse of dimensionality. This is about the limited number of dimensions their grids can hold effectively, and therefore the number of risk factors that the pricing model can actually include. For example, in the attempt to provide a more realistic representation of the firm’s value behaviour in the CB context, Bermúdez and Webber (2004) adopt a firm value approach in which they assume the arrival of a single jump with fixed (non-random) jump size; thereafter the firm value is assumed to evolve as a pure diffusion. Although this assumption facilitates the implementation of the numerical scheme, it still remains simplistic and inadequate to the effective modelling of credit risk, as we discuss in Section II.B. In this respect, Monte Carlo simulation turns out to be the preferred alternative when multiple state variables (especially more than two) and Bermudan features are considered. For example, the adaptation of Monte Carlo
methods by Longstaff and Schwartz (2001) to accommodate early-exercise features is based on
the approximation of the continuation value by a linear combination of suitably chosen ba-
sis functions, and the estimation of the corresponding coefficients by regression. Nevertheless,
Broadie and Detemple (2004) argue that results converge slowly, demanding an increasing num-
ber of basis functions and simulation runs. In the case of CBs, additional care is required on
splitting the spatial domain beforehand into regions where the CB behaves differently (likely
to be called/put/continue existing), otherwise unnecessary approximation of the continuation
value over the unified domain is anticipated to be poor (see Lvov et al. (2004) for a more de-
tailed discussion of this point). Finally, Monte Carlo methods suffer of slow and non-monotone
convergence, preventing the application of convergence-accelerating techniques like Richardson
extrapolation.

Another class of numerical pricing approaches for options, which has become popular in
the most recent literature, is represented by numerical integration methods. These include the
quadrature methods (see Andricopoulos et al. (2003), (2007)) and the Fourier transform-based
methods (see, for example, Carr and Madan (1999) and references therein). In particular,
these two techniques have been successfully combined by Lord et al. (2008) to price Bermudan
vanilla options on a single asset, assumed to evolve as an exponential Lévy process, and a bas-
ket of assets with correlated lognormal trajectories. Fang and Oosterlee (2008b) additionally
deal with the valuation of discretely sampled barrier options using backward convolution en-
hanced by a Fourier-Cosine series expansion, whilst Černý and Kyriakou (2010) price discretely
sampled arithmetic Asian options supported by fractional Fourier transforms (known as chirp
$z$-transforms in the field of signal processing) in a reduced one-dimensional Markovian state
space. An alternative highly efficient method for barrier, lookback and Bermudan options from
Feng and Linetsky (2008) and Feng and Lin (2009) is based on Hilbert transforms. A review of
other transform-based valuation methods for derivative contracts with path-dependence and/or
early-exercise can be found in Fang and Oosterlee (2008a), (2008b).

In the light of the previous discussion, the contribution this paper offers to the current state
of the literature on CBs is threefold. In first place, this paper proposes a Fourier transform pricing technique built on martingale theory, which aims at handling effectively any real-world CB specification, including discrete cash flows, and conversion which either is forced by a call on notice from the issuer, or takes place voluntarily at the holders’ choice before a dividend payment. The method belongs to the class of backward price convolution algorithms, as in Lord et al. (2008), and Černý and Kyriakou (2010). In general terms, the approach proposed in this paper works by evaluating the convertible bond going backwards from maturity, while allowing for the early-exercise features and discrete payments at relevant time points. Secondly, this paper uses a market model comprising four risk factors: an underlying evolving as a diffusion augmented by jumps, subject to random arrival and jump size, and stochastic interest rates. We consider both the cases of the Gaussian jump diffusion process (Merton (1976), Bates (1991)) and the Kou process (Kou (2002)). To the best of our knowledge, such a setup has not been implemented earlier in the convertible bonds’ literature due to dimensionality issues. The numerical pricing scheme proposed in this paper is shown to be flexible enough to handle the dimensionality imposed by the abovementioned market model, while remaining smoothly convergent and precise. Finally, we show that the bivariate log-firm value-interest rate process falls within the class of affine models in the sense of Duffie et al. (2003) (see Kallsen (2006) for details) and, using the results on numéraire pair changes from Geman et al. (1995), we derive its characteristic function in closed form. Characterizing the law of the underlying model is pivotal for the implementation of the convolution algorithm.

The proposed pricing methodology is then tested using different parametrization of the market framework adopted in this paper. In particular, we examine the discrepancy between the prices generated by the two jump diffusion models under consideration as function of the model parameter values and the moneyness (measure of the likelihood of conversion) of the convertible bond. We explore the effects of coupons payable to the CB holders and dividends distributed to the current stock holders, as well as the impact of varying call policy on the computed prices.
The remaining of the paper is organized as follows. In Section II we introduce the basic notation and our assumptions for the firm value and interest rate processes. We then justify our choice based on the empirical evidence available on the credit-spread term structure, and provide intuition on how to overcome some significant impracticalities related to the calibration of the firm value model. In Section III we describe the CB design under consideration, with particular emphasis on the optimal call strategy assumed for the issuing firm; we also derive the payoff to the CB holders after a with-notice call by the firm. In Section IV, we develop the theoretical ground for the Fourier transform-based backward pricing convolution scheme and discuss its implementation via discrete Fourier transform. Section V demonstrates the proposed numerical scheme in practice. Section VI concludes.

II Market model

From a valuation perspective, a pricing model for CBs requires assumptions on the term structure of interest rates, the dynamic followed by the asset underlying the conversion option, and the firm’s default-driving mechanism. In this paper we adopt a structural approach to model the underlying of the contract and the default-triggering event; in particular, we assume that the dynamic of the firm value is driven by a jump diffusion process. The detailed assumptions of our model are presented in the following sections, together with the rationale of our choice. Finally, we use the Vašíček (1977) model for the term structure of interest rates.

A A unified firm value-interest rate setup

Let \( (\Omega, \mathcal{F}, \mathbb{P} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a complete filtered probability space, where \( \mathbb{P} \) is some risk-neutral probability measure. We assume that the firm value \( V \) is given by

\[
V_t = e^{Y_t},
\]
for a jump diffusion process

\begin{equation}
Y_t = Y_0 + \int_0^t \left( r_s - \frac{\sigma^2}{2} - \lambda (\phi_L (-i) - 1) \right) ds + \sigma W_t + \int_{\mathbb{R}} lN_t (dl),
\end{equation}

where $Y_0 = \ln V_0$, $W$ is a $\mathbb{F}$-adapted standard Brownian motion in $\mathbb{R}$, $N$ is a time-homogeneous Poisson process with constant intensity $\lambda$ and $L$ is the random jump size, which is modelled by a sequence of independent and identically distributed random variables with $\mathbb{E}(L) = \mu_L$, $\mathbb{V}ar(L) = \sigma^2_L$ and characteristic function $\phi_L$; the processes $W$ and $N$ and the random variable $L$ are assumed to be mutually independent. As far as the distribution governing $L$ is concerned, two popular choices in the literature are the double exponential distribution (Kou (2002)) and the normal distribution (Merton (1976), Bates (1991)). Specifically, in the case of the double exponential jump diffusion process (DEJD), $L$ has characteristic function

\[ \phi_L (u) = \frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu}, \]

where $p, q \geq 0$, $\eta_1 > 1$, $\eta_2 > 0$, and $p + q = 1$ as they represent the (risk-neutral) probabilities of an upward and a downward jump respectively. In the case of the Merton jump diffusion model (MJD), instead, we assume that $L$ follows a normal distribution; therefore the characteristic function is

\[ \phi_L (u) = e^{i\mu_L u - \frac{\sigma^2_L u^2}{2}}. \]

The short rate process $r$ is assumed to evolve according to the Vašíček (1977) model; hence, the log-price $\ln P_t (v)$ at $t > 0$ of a pure-discount bond maturing at $v \geq t$ satisfies

\begin{align*}
(2) & \quad \ln P_t (v) = \ln P_0 (v) + \int_0^t \left( r_s - \frac{m^2_s (v)}{2} \right) ds + \int_0^t m_s (v) dW_{r,s}, \\
(3) & \quad |m_t (v)| = \frac{\sigma_r}{\kappa} \left( 1 - e^{-\kappa (v-t)} \right), \quad \kappa, \sigma_r > 0,
\end{align*}

where $W_r$ is a standard Brownian motion, such that $W$ and $W_r$ have constant correlation $\rho$. 
whereas $W_t$ is independent of both $N$ and $L$. Alternatively,

\begin{align*}
\ln P_t(v) &= A_t(v) - B_t(v) r_t; \\
A_t(v) &= \frac{1}{\kappa^2} \left( B_t(v) - v + t \right) \left( \mu_r \kappa^2 - \frac{\sigma_r^2}{2} \right) - \frac{\sigma_r^2 B_t^2(v)}{4\kappa}, \mu_r > 0, \\
B_t(v) &= \frac{1}{\kappa} \left( 1 - e^{-\kappa(v-t)} \right).
\end{align*}

The results (2)-(6) can be found, for example, in Hull (2003) and Vašíček (1977).

**B Stock versus Firm value and real-world considerations**

Generally speaking, the available approaches to model credit risk can be classified in two main categories: the structural and intensity-based (reduced-form) models\(^1\).

The main feature of the structural methods is the fact that the credit events are triggered by movements of the firm value below some boundary. Thus, a key aspect of this framework is the modelling of the firm value process. Structural default has been first introduced by Merton (1974), who considers the possibility of bankruptcy only at maturity of the risky bond. Various modified versions of the original Merton (1974) model have been proposed, including Black and Cox (1976), Longstaff and Schwartz (1995) (with stochastic interest rate), Leland (1994) and Leland and Toft (1996), amongst others. The main difference between these methods and Merton (1974) is the inclusion of a stopping time, which represents the default time upon the breaching of the benchmark level by the firm value trajectory at any time over the term of the contract. In the CB context, Ingersoll (1977a), (1977b) follows Merton (1974), while Brennan and Schwartz (1977), (1980) proceed one step further by additionally allowing for default prior to maturity.

All the abovementioned contributions use a drifted Brownian motion to describe the dynamic of the log-firm value, in the spirit of Black and Scholes (1973) and Merton (1973). Nevertheless,\(^1\)

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\(^1\)Apart from the purely structural and intensity-based models, there are some hybrid approaches which combine elements from both techniques. More about these can be found in Bielecki and Rutkowski (2002).
by nature of the diffusion process, this particular assumption precludes a sudden, unexpected drop of the firm value process below the default-triggering threshold level. Consequently, the firm can go bankrupt only when its value reaches exactly that level after a smooth decline. According to Zhou (1997), for a firm which is subject to such a “predictable” default and is not in financial distress, the probability of default in the short-run is negligible, although the credit risk becomes more significant for longer maturities. Therefore, these models imply a flat term structure of credit spread at zero level for short maturities with an increasing slope at longer maturities. Unfortunately, such a shape for the credit-spread curve is inconsistent with the empirical results of Fons (1994) and others. According to these contributions, the curve for certain corporate bonds may be observed to be not only upwards-sloping, but also flat or even downwards-sloping. A possible route to face this matter of “predictability” is to include unforeseeable jumps into the dynamics of the firm value evolution, using for example Poisson jumps, as in Zhou (1997), Hilberink and Rogers (2002), Chen and Kou (2005) and Dao and Jeanblanc (2006), or by completely discarding the diffusion component and replacing it with pure jumps, as in Madan (2000). In both cases, bankruptcy takes place in the form of a jump, i.e., by crossing the critical default boundary without exactly touching it. As Zhou (1997) points out, in a jump diffusion structural approach, the diffusion component generates conceptual insights on default behaviour, since the default events can be associated to the smooth decline of the firm’s capital structure, whilst the additional existence of jumps allows for likely external impacts and enables a more flexible fitting to the observed credit spreads. Another way to produce high short-term spreads is by incorporating a stochastic barrier level, as in the CreditGrades (2002) technical document; this feature proves to raise the likelihood for the firm’s assets being at a level which is closer to the bankruptcy point than otherwise believed. In the CB context, we note that Bermúdez and Webber (2004) resort to the firm value technique by implementing a jump-augmented geometric Brownian motion, where the exogenous default event coincides with the jump time of a time-inhomogeneous Poisson counter with “semi-stochastic” intensity (see Lando (1998)).
The alternative approach to modelling default is known as the intensity-based technique. The distinguishing feature of this framework is the unpredictability of the default time, which is totally inaccessible (i.e., it comes as a “surprise”). Such a default is said to be exogenous, exactly because it occurs in a sudden manner and is related to an external cause. The concept behind intensity-based default is simple: the instantaneous probability of default is exogenously specified by means of some intensity (hazard rate), which may be treated either “semi-stochastically”, as a function of the underlying stock, or directly stochastically. In the CB context, the reduced-form technique of Duffie and Singleton (1999) is mostly popular (see, for example, Takahashi et al. (2001), Davis and Lischka (2002), Andersen and Buffum (2003) and Carayannopoulos and Kalimipalli (2003)). According to this specification, pre-default prices can be reasonably assumed to be driven by a diffusion process. It can be argued though that these (semi-) stochastic intensity-based models unnecessarily penalize the default-free equity component of the convertible bond, as the default intensity appears in the drift part of the stock process. In general, a company’s ability to issue stock is not strongly influenced by its credit rating and it can always deliver that stock. To the contrary, coupon and principal payments depend on the issuer’s timely access to the required amounts. Inability to access these payments at the right time induces credit risk. On the same grounds, Tsiveriotis and Fernandes (1998) choose to split the CB into artificial debt-only and equity-only elements. The debt-only part is discounted at a higher rate, subject to a constant spread over the short rate, to reflect the default risk associated to it. Then, the two components are added to provide the overall CB price\textsuperscript{2}. On the other hand, Takahashi et al. (2001) claim that the assumption of the stock not being subject to default risk is likely to result into model inconsistency with the market. Furthermore, Takahashi et al. (2001), Davis and Lischka (2002) and Carayannopoulos and Kalimipalli (2003) presume that, upon default, the stock instantaneously jumps to zero. Based on empirical results, Ayache et al. (2003) consider this assumption as extreme and controversial. For this reason, they apply a

\textsuperscript{2}The early works of McConnell and Schwartz (1986), Cheung and Nelken (1994) and Ho and Pfeffer (1996) also consider a constant credit spread-adjusted discount rate, which is applicable, however, to the entire CB.
proportional reduction to the pre-default stock price at the time of default; however, the optimal choice of this reduction is likely to be another vulnerable point of the stock-based models.

In the light of the previous discussion, in this note we follow Bermúdez and Webber (2004) and adopt a firm value approach to credit risk in order to avoid a disputable treatment of equity. Nevertheless, we emphasize on the simple assumption of their approach that default occurs only once; thereafter the firm value is assumed to evolve as a pure diffusion and, hence, any possibility of future exogenous default events to occur is eliminated. This compromise is necessary by limitations of the PDI numerical scheme they employ. Here, however, we manage to overcome this modelling weakness and adopt an exponential jump diffusion firm value approach, as in Merton (1976) and Kou (2002), so that default can be reached following a number of consecutive shocks in the value of the firm. To the best of our knowledge, this is the first time that these two jump diffusion processes are utilized in the context of CBs’ valuation.

C Calibration issues

Despite its appealing implications in the credit risk context, a model based on the value of the firm, which is not directly market-observable, traditionally suffers in calibration. The lack of this information poses crucial impracticalities especially in an incomplete market, such as the one proposed in this paper, which we need to handle as efficiently as possible. As King (1986) explains, many of the firm’s liabilities are not traded in organized exchanges or have limited trading activity, as opposed to the highly liquid stock, prohibiting their synchronous observation in many instances and, hence, their simultaneous estimation. The first contributions offering a solution to the estimation problem include Carayannopoulos (1996), who suggests the volatility of the common stock (obtained from the market) as proxy for the volatility of the firm value (the former actually forms an upper bound to the latter) and King (1986), who proposes a leverage-adjusted stock volatility for the firm value. As a consequence, in the case of Carayannopoulos (1996), some overpricing effects have been reported, especially for deep in-the-money CBs, due
to the overstated firm value volatility, while King’s (1986) version appears to work even less efficiently. A more recent candidate for the firm’s assets volatility is the one which recovers, as consistently as possible, the market CDS (Credit Default Swap) spread on that firm (see, for instance, CreditGrades (2002)).

Although our intention here is not to actually calibrate the firm value and interest rate processes, we brief on alternative promising calibration routes, whose application is postponed to a future stage of our research.

Starting with the short rate, calibration can be operated, as described in Barone-Adesi et al. (2003), by adopting the Hull and White’s (1990) generalization of the original Vašíček (1977) model, with asymptotic mean level $\mu_r$ chosen in accordance with the market zero-coupon bond prices (see equation (5)), as at a given reference date. The remaining parameters, $\kappa$ and $\sigma_r$, are obtained from the minimization of the error between theoretical and market prices of liquid caps, as at the reference date.

As far as the firm value is concerned, the novel non-parametric calibration methodology by Cont and Tankov (2004a) for Lévy processes guarantees consistency with observed market prices, via the minimization of a well-defined model-market prices distance functional. Further, it allows for dependence on the information gained since the previous calibration, via an entropic measure of the closeness between the current market martingale measure and the prior measure (the outcome from the previous calibration). Cont and Tankov (2004a) test successfully the performance of their algorithm on European plain vanilla options on stocks in a DEJD setup. An extensive discussion on this procedure and the associated technicalities can be found in Cont and Tankov (2004a), (2004b). Therefore, instead of seeking to infer the unknown firm parameter values from stock data, we may use the information contained in the historical prices of ordinary and convertible bonds from the same issuer. However, because of our model’s structural nature, a complication arises due to the need to infer simultaneously all claims. To eliminate this complication, we should ideally restrict our sample to firms with simple capital structures consisting only of common stock, senior debt and subordinated convertible debt. Then, based
on the derived guesses for the parameters, we could test the out-of-sample forecasting power
of our CB model. Zabolotnyuk et al. (2007) have set up the structural model of Brennan and
Schwartz (1977), (1980), where riskless senior debt has been easily incorporated to be deduced
as part of the calibration procedure. In this way, they have managed to calibrate effectively
and produce price forecasts, which are comparable to the Tsiveriotis and Fernandes (1998)
stock-based model predictions for the same sample of firms.

III Convertible bonds: contract features

A convertible bond is an ordinary bond which additionally offers the investors the option to
exchange it for a predetermined number of shares at certain points in time. In this respect,
the conversion rights originate a Bermudan option. In the case of conversion, each investor
receives the conversion value $\gamma V_t$, where $\gamma \in (0, 1)$ denotes the dilution factor, i.e., the fraction
of common stock possessed by each CB holder post-conversion, which is usually constant over
time, whilst $V_t$ is the aggregate market value of the firm’s outstanding securities including the
convertible bonds (firm value). The CB issue usually offers regular aggregate coupon payments
$C_{t_j}$ at time $t_j \in [0, T]$ and, for $m$ outstanding CBs, this corresponds to $c_{t_j} = C_{t_j}/m$ payment
per bond. In the case the issue is kept alive to its expiration at time $T$, it is redeemed for a
total face value $mF$. The firm’s stock holders receive, instead, a discrete aggregate dividend
$D_{t_i}$ at the dividend date $t_i \in [0, T]$, such that $t_j \neq t_i$.

Further, CBs also contain a call option allowing the issuer to redeem it prematurely in
exchange for the current call price; the issuer is in general obliged to announce his/her decision
to call the bond a certain period in advance (call notice period). Once the CB is called, the
investor needs to consider if it is the case to exercise the conversion option at the end of the
call notice period, in order to convert instead of receiving the call price. Further, put option
provisions entitling the investor to force a premature repurchase of the CBs by the issuing firm,
are another feature which is sometimes met. However, in this paper we ignore the putability
provision, as it has been shown to cause minor effects on the CB’s price (see, for example, Bermúdez and Webber (2004)).

The existence of a callability provision implies that the CB’s payoff depends on the optimal exercise strategy adopted by the issuer. This is discussed in the next section.

A The optimal call strategy

Under the assumption of a market not subject to any imperfections, in which the Modigliani-Miller (1958) theorem holds and no call notice applies, Ingersoll (1977a) proves that the optimal call policy for a callable convertible issue is to call as soon as the firm value $V_t$ reaches the critical level $K_t/\gamma$, for a deterministic call price $K_t$, which usually is either fixed by the firm at the issue of the contract, or a piecewise constant function (see Ammann et al. (2008)). This feature endows the CB with path-dependence and, consequently, implies the need for frequent monitoring. Despite his original result, Ingersoll (1977b) observes empirically that firms tend to follow different call strategies; they choose, in fact, to call when the conversion value is in excess of the call price. Forcing conversion by a call at the earliest opportunity, instead, leads to undervalued CBs, as shown in Carayannopoulos (1996) and Carayannopoulos and Kalimipalli (2003).

The extensive empirical analysis carried out by Asquith and Mullins (1991) and Asquith (1995) shows that the observed call delays can be attributed mainly to three factors: a call notice period, the existence of significant cash flows advantages, and a safety premium on the given call price. In details, the call notice period, which prohibits the CB to be called for as long as this period is active, in fact proves to be the main reason for the delayed calls; for those CBs that are not called at the end of this period, the firm might be saving cash by delaying the call if, for example, the after-corporate tax coupons on the CB are less than the dividends payable post-conversion. Another important reason for delaying is linked to the existence of a safety premium imposed by the issuing firm prior to the call announcement, in the attempt to
guarantee that the conversion value will still exceed the call price at the end of the call notice period and, hence, avoid the bond redemption in cash.

In this work, we build on these findings and formulate the optimal call policy for the CB as follows. Let $\vartheta \in (0,1)$ denote the safety premium mentioned above; then, the firm’s optimal call announcement is given by the stopping time

$$
\tau^c = \inf \left\{ t : V_t \geq \frac{(1 + \vartheta)K_t}{\gamma} \right\}.
$$

Assume the call notice period is $s^c$ and define the accrued interest $AccIR = t_{j+1} - t_j c_{t_{j+1}}$, $t_j \leq \tau^c + s^c < t_{j+1}$, such that the call price at the end of this period is $K_{\tau^c + s^c} = K_{\tau^c} + AccIR$. Then, the investor’s payoff upon the call of the CB by the issuer is $\bar{K}_{\tau^c + s^c} (V_{\tau^c + s^c}) = \max(\gamma V_{\tau^c + s^c}, K_{\tau^c + s^c})$, and its no-arbitrage price at the time of the call is

(7) $$
\tilde{K}_{\tau^c} (V_{\tau^c}, r_{\tau^c}) = \mathbb{E} \left( e^{-\int_{\tau^c}^{\tau^c+s^c} r_s ds} \bar{K}_{\tau^c + s^c} (V_{\tau^c + s^c}) \bigg\vert \mathcal{F}_{\tau^c} \right).
$$

B The payoff function and pricing considerations

Because of the early-exercise rights embedded in the CB, we define the contract payoff (per bond) function $\tilde{H}_t$ at any possible decision time $t \in (0, T]$ as follows.

At maturity $T$, the investors can choose between converting to common stock (see Brennan and Schwartz (1977), Lemma 1) or receiving the face value and the last coupon, providing that the firm can afford the total of this payment. Otherwise, they recover the outstanding firm value at that time. Hence,

(8) $$
\tilde{H}_T (V_T, r_T) = \begin{cases} 
\gamma V_T, & V_T \geq (F + c_T)/\gamma \\
F + c_T, & mF + C_T \leq V_T < (F + c_T)/\gamma \\
V_T/m, & V_T < mF + C_T
\end{cases}.
$$

At a date where no coupon or dividend payment are due, the CB may be forced by a call to
conversion, or continue to exist till at least the next monitoring point, i.e.,

$$ \tilde{H}_t(V_t, r_t) = \begin{cases} 
\tilde{K}_t(V_t, r_t), & V_t \geq \frac{(1+\vartheta)K_t}{\gamma}, \quad 0 < t < T, \ t \neq t_i, t_j \\
H_t(V_t, r_t), & V_t < \frac{(1+\vartheta)K_t}{\gamma}, \quad 0 < t < T, \ t \neq t_i, t_j 
\end{cases}, $$

where $H_t$ denotes the no-arbitrage (continuation) value of the CB, and $\tilde{K}_t$ is given by equation (7). At a coupon date, $t_j$, the payoff of the CB depends on whether the firm has enough funding to meet the claim. If $V_{t_{j-}} \leq C_{t_j}$, the CB defaults, its value is $H_{t_j} = 0$, since $0 \leq H_{t_j} \leq V_{t_j}$ by limited liability and the Modigliani-Miller (1958) theorem, and $C_{t_j} = V_{t_{j-}}$, i.e., the investor sizes the available assets. If, instead, $V_{t_{j-}} > C_{t_j}$ and for as long as the CB is uncalled, the contract remains in force and the coupon is paid in full. On the other hand, if the CB is called, its holders receive both the call payoff and the coupon. Hence,

$$ \tilde{H}_{t_{j-}}(V_{t_{j-}}, r_t) = \begin{cases} 
\frac{V_{t_{j-}}}{m}, & V_{t_{j-}} \leq C_t, \quad 0 < t < T, \ t = t_j \\
H_t(V_t, r_t) + c_t, & C_t < V_{t_{j-}} < \frac{(1+\vartheta)K_t}{\gamma}, \quad 0 < t < T, \ t = t_j \cdot \\
\tilde{K}_t(V_t, r_t) + c_t, & V_{t_{j-}} \geq \frac{(1+\vartheta)K_t}{\gamma}, \quad 0 < t < T, \ t = t_j 
\end{cases}, $$

Finally, at a dividend date, $t_i$, the investors may find optimal to convert prior to the dividend payment\(^3\) (voluntary conversion). The following condition, which is proved in Brennan and Schwartz ((1977), Lemma 1), then applies:

$$ \tilde{H}_{t_{i-}}(V_{t_{i-}}, r_t) = \max \left( H_t(V_t, r_t), \gamma V_{t_{i-}} \right), \quad 0 < t < T, \ t = t_i. $$

The payoff function defined by equations (8)-(11) highlights the Bermudan style of the CB; this feature and the high path-dependency induced by the callability provision imply that the no-arbitrage price of the CB, $H_0$, can be recovered only by numerical approximation.

\(^3\)At a dividend date, the existing stock holders are entitled to receive dividends for as long as the firm can afford their payment, providing that it has already met all the other claims ranking above them.
IV The Fourier transform-based pricing algorithm

Fourier analysis has been successfully used for the pricing problem of American and Bermudan vanilla options by Lord et al. (2008), who apply the transform with respect to the log-spot price in a backward recursive scheme. Their method relies on the property of independent increments shown by the log-returns in their market model. In this paper, we adapt this approach to the pricing of CBs; however, the straightforward extension of the method is not possible due to the fact that in our model the increments of the log-firm value are not independent (see equation (1)). Further, the contract under consideration presents a higher degree of complexity due to the presence of intermediate discrete payments, exotic features, like call provision with attached call notice, and additional risk factors, such as stochastic interest rates.

Thus, we consider the partition $\mathcal{T} = \{t_k\}_{k=0}^{n}$, $n \in \mathbb{N}$ of the contract’s term $[0, T]$ representing the set of the decision dates. Further, we assume for ease of exposition that these dates are equally spaced so that $t_k - t_{k-1} = \delta t$ for $0 < k \leq n$, with $t_0 = 0$, $t_n = T$. With these assumptions in mind, the price of the CB is the solution to a dynamic programming problem as stated in the following.

**Theorem 1** Consider $A_{t_{k-1}}(t_k)$, $B_{t_{k-1}}(t_k)$, $P_{t_{k-1}}(t_k)$ and $\kappa > 0$ as in (4)-(6), $0 < k \leq n$. Define the functions $g_k$, $g_{r,k}$ as

$$g_{k-1} (y, y_r) = \begin{cases} 
\ln (e^y - \bar{D}_{t_{k-1}}) - A_{t_{k-1}}(t_k) + B_{t_{k-1}}(t_k) y_r, & 1 \leq k \leq n, \ k - 1 = i, \\
\ln (e^y - C_{t_{k-1}}) - A_{t_{k-1}}(t_k) + B_{t_{k-1}}(t_k) y_r, & 1 \leq k \leq n, \ k - 1 = j, \ y > \ln C_{t_{k-1}}, \\
y - A_{t_{k-1}}(t_k) + B_{t_{k-1}}(t_k) y_r, & 1 \leq k \leq n, \ k - 1 \neq i, j,
\end{cases}$$

$$g_{r,k-1} (y_r) = y_r e^{-\kappa (t_k - t_{k-1})}, \ 1 \leq k \leq n.$$  

\footnote{For completeness, impose time-subscripts $t_0 = t_0$, $t_n = t_n$.}
such that the pairs

\begin{equation}
(Z_{k-1}, Z_{r,k-1}) = \left( Y_{t_{k-1}} - g_{k-1} \left( Y_{t+k-1}, r_{tk-1} \right), r_{tk} - g_{r,k-1} \left( r_{tk-1} \right) \right), \quad 1 \leq k \leq n,
\end{equation}

have a forward-risk-adjusted joint density function $f^*$, for all $k$.\footnote{As shown in Appendix C, equation \text{(A.15)}, the pair $(Z, Z_r)$ forms a sequence of identically distributed random variables; hence, we may drop the time-subscripts from $(Z_{k-1}, Z_{r,k-1})$, as well as from the associated quantities.} Given the CB value function $H_k$ and the payoff functions $\tilde{H}_k$ as in \textbf{(8)-(11)}, we define the auxiliary function

\begin{equation}
\tilde{H}_{k-1} (x, x_r) = \int_{\mathbb{R} \times \mathbb{R}} \tilde{H}_{k-1} (x + z, x_r + z_r) f^* (z, z_r) \, d(z, z_r)
\end{equation}

\begin{equation}
= \tilde{H}_{k-1} * f^* (-z, -z_r), \quad 1 \leq k \leq n.
\end{equation}

Then, the no-arbitrage price of the CB at each decision date is

\begin{equation}
H_{k-1} \left( Y_{t_{k-1}}, r_{tk-1} \right) = P_{t_{k-1}} \left( t_k \right) \tilde{H}_{k-1} \left( g_{k-1} \left( Y_{t+k-1}, r_{tk-1} \right), g_{r,k-1} \left( r_{tk-1} \right) \right), \quad 1 \leq k \leq n.
\end{equation}

The price of the CB at inception is therefore $H_0 \left( Y_{t_0}, r_{t_0} \right)$.

\textbf{Proof.} Based on the first fundamental theorem of asset pricing (see Delbaen and Schachermayer (1994)), Geman et al. (1995), Corollary 2) and the change to the forward-risk-adjusted probability measure $\mathbb{P}^*$, as described in Proposition 2 (see Appendix B), for $1 \leq k \leq n$ we obtain

\begin{equation}
H_{k-1} \left( Y_{t_{k-1}}, r_{tk-1} \right) = \mathbb{E} \left( e^{-\int_{t_{k-1}}^{t_k} r_s \, ds} \tilde{H}_{k-1} \left( Y_{t_{k-1}}, r_{tk} \right) \left| \mathcal{F}_{t_{k-1}} \right. \right) = P_{t_{k-1}} \left( t_k \right) \mathbb{E}^* \left( \tilde{H}_{k-1} \left( Y_{t_{k-1}}, r_{tk} \right) \left| \mathcal{F}_{t_{k-1}} \right. \right).
\end{equation}
From (12),
\[
\mathbb{E}^* \left( \tilde{H}_{k-} \left( Y_{t_k}, r_{t_k} \right) \bigg| \mathcal{F}_{t_k-1} \right) = \mathbb{E}^* \left( \tilde{H}_{k-} \left( g_{k-1-} \left( Y_{t_{k-1}}, r_{t_{k-1}} \right) + Z_{k-1-} \cdot g_{r,k-1} \left( r_{t_{k-1}} \right) + Z_{r,k-1} \right) \bigg| \mathcal{F}_{t_k-1} \right) \\
= \int_{\mathbb{R} \times \mathbb{R}} \tilde{H}_{k-} \left( g_{k-1-} \left( Y_{t_{k-1}}, r_{t_{k-1}} \right) + z, g_{r,k-1} \left( r_{t_{k-1}} \right) + z_r \right) f^* \left( z, z_r \right) d \left( z, z_r \right) \\
= \tilde{H}_{k-1-} \left( g_{k-1-} \left( Y_{t_{k-1}}, r_{t_{k-1}} \right), g_{r,k-1} \left( r_{t_{k-1}} \right) \right),
\]
as defined in (13). Hence,
\[
H_{k-1-} \left( Y_{t_{k-1}}, r_{t_{k-1}} \right) = P_{t_k-1} \left( t_k \right) \tilde{H}_{k-1-} \left( g_{k-1-} \left( Y_{t_{k-1}}, r_{t_{k-1}} \right), g_{r,k-1} \left( r_{t_{k-1}} \right) \right); \\
\]
this result is then used to compute the new payoff \( \tilde{H}_{k-1-} \left( Y_{t_{k-1}}, r_{t_{k-1}} \right) \), in accordance with (8)-(11), to be applied in the subsequent iteration.

The dynamic programming problem described by (8)-(11) and (13)-(14) is a successive application of the valuation procedure under the forward-risk-adjusted measure. Therefore, by backward induction, we eventually obtain the convertible bond price at \( t_0, H_0 \left( Y_{t_0}, r_{t_0} \right) \).

The implementation of the procedure set in Theorem 1 requires the computation of the payoff upon call of the CB by the issuing firm. Based on the optimal call strategy set in Section III.A, this is given in the following.

**Theorem 2** Consider \( A_{\tau^c} (\tau^c + s^c) \), \( B_{\tau^c} (\tau^c + s^c) \) and \( P_{\tau^c} (\tau^c + s^c) \) as in (4)-(6) and define the function \( h_{\tau^c} \) as
\[
h_{\tau^c} (y, y_r) = \begin{cases} 
  y - A_{\tau^c} (\tau^c + s^c) + B_{\tau^c} (\tau^c + s^c) y_r, & \tau^c \neq t_{j-} \\
  \ln (e^y - C_{\tau^c}) - A_{\tau^c} (\tau^c + s^c) + B_{\tau^c} (\tau^c + s^c) y_r, & \tau^c = t_{j-}
\end{cases},
\]
such that the random variable
\[
Z_{\tau^c} \doteq Y_{\tau^c+s^c} - h_{\tau^c} (Y_{\tau^c}, r_{\tau^c})
\]
has a forward-risk-adjusted density function $\zeta^*$. Recall from Section III.A the definition of the function $\bar{K}_t$

$$\bar{K}_{\tau + s^c} (y) = \max (\gamma e^y, K_{\tau + s^c})$$

and further define the function $\hat{K}_t$ as

$$(15) \quad \hat{K}_{\tau^c} (x) = \int_{\mathbb{R}} \bar{K}_{\tau + s^c} (x + z) \zeta^* (z) \, dz = \bar{K}_{\tau + s^c} * \zeta^* (-z).$$

Then, the payoff $\tilde{K}_t$ at the announcement date of the exercise of the call rights by the issuing firm is

$$(16) \quad \tilde{K}_{\tau^c} (Y_{\tau^c}, r_{\tau^c}) = P_{\tau^c} (\tau^c + s^c) \hat{K}_{\tau^c} (h_{\tau^c} (Y_{\tau^c}, r_{\tau^c})).$$

**Proof.** Based on the same argument as in the previous proof, we obtain from equation (7)

$$\tilde{K}_{\tau^c} (Y_{\tau^c}, r_{\tau^c}) = \mathbb{E} \left( e^{-\int_{\tau^c}^{\tau^c+s^c} r_{s^c} \, ds} \bar{K}_{\tau + s^c} (Y_{\tau + s^c}) \mid \mathcal{F}_{\tau^c} \right) = P_{\tau^c} (\tau^c + s^c) \mathbb{E}^* \left( \bar{K}_{\tau + s^c} (Y_{\tau + s^c}) \mid \mathcal{F}_{\tau^c} \right).$$

Moreover,

$$\mathbb{E}^* \left( \bar{K}_{\tau^c + s^c} (Y_{\tau^c + s^c}) \mid \mathcal{F}_{\tau^c} \right) = \mathbb{E}^* \left( \bar{K}_{\tau^c + s^c} (h_{\tau^c} (Y_{\tau^c}, r_{\tau^c}) + Z_{\tau^c}) \mid \mathcal{F}_{\tau^c} \right)$$

$$= \int_{\mathbb{R}} \bar{K}_{\tau^c + s^c} (h_{\tau^c} (Y_{\tau^c}, r_{\tau^c}) + z) \zeta^* (z) \, dz$$

$$= \hat{K}_{\tau^c} (h_{\tau^c} (Y_{\tau^c}, r_{\tau^c})).$$

Therefore,

$$\tilde{K}_{\tau^c} (Y_{\tau^c}, r_{\tau^c}) = P_{\tau^c} (\tau^c + s^c) \hat{K}_{\tau^c} (h_{\tau^c} (Y_{\tau^c}, r_{\tau^c})), $$

which concludes the proof. ■

The CB pricing problem stated by equations (13)-(14) and the associated optimal call strategy problem given in equations (15)-(16) can be solved by backward recursion using Fourier
inversion techniques. This, though, requires the knowledge of the characteristic functions of the random quantities involved, i.e., the pair \((Z, Z_r)\) for problem (13)-(14) and \(Z\) for problem (15)-(16). These results are given in the following.

**Proposition 1** Consider the forward-risk-adjusted density functions \(\zeta^*, f^*\) and the random variables \(Z, Z_r\) defined in Theorems 7 and 3. We define the forward-risk-adjusted characteristic functions \(\varphi^*\) and \(\phi^*\) of \(Z\) and \((Z, Z_r)\) respectively,

\[
\varphi^*(u) = \mathbb{E}^* ( e^{iuZ} ) = \int_{\mathbb{R}} e^{iuz} \zeta^*(z) \, dz \tag{17}
\]

and

\[
\phi^*(u, u_r) = \mathbb{E}^* ( e^{iuZ+iu_rZ_r} ) = \int_{\mathbb{R} \times \mathbb{R}} e^{iuz+iu_rz_r} f^*(z, z_r) \, d(z, z_r). \tag{18}
\]

(17) and (18) are available through the closed analytical form (A.15).

**Proof.** See Appendix C. \(\blacksquare\)

Hence, the pricing problem (13)-(14) can be solved at each time step by moving backwards from maturity, after transforming to the Fourier domain and, then, re-expressing in the form of Fourier-inversion integrals. In particular,

\[
\mathcal{F}(\hat{H}_{k-1}) = \mathcal{F}(\bar{H}_{k-1} \ast f^*(-z, -z_r)) = \mathcal{F}(\bar{H}_{k-1}) \mathcal{F}(f^*(-z, -z_r)) = \mathcal{F}(\bar{H}_{k-1}) \bar{\phi}^*, \tag{19}
\]

\[
\hat{H}_{k-1} = P_{t_k-1}(t_k) \mathcal{F}^{-1}(\mathcal{F}(\bar{H}_{k-1}) \bar{\phi}^*), \tag{20}
\]

where \(\mathcal{F}\) denotes the Fourier transform (see Appendix A) and \(\bar{\phi}^*\) the complex conjugate of \(\phi^*\).

In analogy to (19)-(20), we compute the solution to (15)-(16)

\[
\mathcal{F}(\hat{K}_{r\epsilon}) = \mathcal{F}(\bar{K}_{r\epsilon + s\epsilon} \ast \zeta^*(-z)) = \mathcal{F}(\bar{K}_{r\epsilon + s\epsilon}) \mathcal{F}(\zeta^*(-z)) = \mathcal{F}(\bar{K}_{r\epsilon + s\epsilon}) \bar{\varphi}^*, \tag{21}
\]

\[
\hat{K}_{r\epsilon} = P_{t\epsilon}(\tau^c + s^c) \mathcal{F}^{-1}(\mathcal{F}(\bar{K}_{r\epsilon + s\epsilon}) \bar{\varphi}^*), \tag{22}
\]

21
where $\varphi^*$ is the complex conjugate of $\varphi^*$.

A Implementation of the algorithm

**Definition 1** Consider the $n_1 \times n_2$ matrix $A = \{a_{m_1m_2}\}_{m_1=0, m_2=0}^{n_1-1, n_2-1}$. The discrete Fourier transform (DFT) of $A$ is the $n_1 \times n_2$ matrix $B = \{b_{j_1j_2}\}_{j_1=0, j_2=0}^{n_1-1, n_2-1}$ satisfying

$$b_{j_1j_2} = \frac{1}{n_1n_2} \sum_{m_1=0}^{n_1-1} \sum_{m_2=0}^{n_2-1} e^{i\frac{2\pi}{n_1}j_1m_1 + i\frac{2\pi}{n_2}j_2m_2} a_{m_1m_2}.$$ 

The inverse discrete Fourier transform of $B$ satisfies

$$a_{m_1m_2} = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} e^{-i\frac{2\pi}{n_1}j_1m_1 - i\frac{2\pi}{n_2}j_2m_2} b_{j_1j_2}.$$ 

The sums in Definition 1 can be computed by means of the inverse FFT and FFT algorithms, which are readily available in MATLAB. In consistency with MATLAB built-in functions, we have that $B = \text{ifft2}(A)$ and $A = \text{fft2}(B)$.

**Definition 2** Consider the uniform grids $(x_1, x_2) \doteq (\{x_{1,0} + m_1\delta x_1\}_{m_1=0}^{n_1-1}, \{x_{2,0} + m_2\delta x_2\}_{m_2=0}^{n_2-1})$ and $(u_1, u_2) \doteq (\{u_{1,0} + j_1\delta u_1\}_{j_1=0}^{n_1-1}, \{u_{2,0} + j_2\delta u_2\}_{j_2=0}^{n_2-1})$. We define the $n_1 \times n_2$ matrices $\bar{A} = \{\bar{a}_{m_1m_2}\}_{m_1=0, m_2=0}^{n_1-1, n_2-1}$ and $\bar{B} = \{\bar{b}_{j_1j_2}\}_{j_1=0, j_2=0}^{n_1-1, n_2-1}$ such that

$$\bar{b}_{j_1j_2} = \sum_{m_1=0}^{n_1-1} \sum_{m_2=0}^{n_2-1} e^{iu_1x_{1,m_1} + iu_2x_{2,m_2}} \bar{a}_{m_1m_2} = e^{i(u_1-u_{1,0})x_{1,0} + (u_2-u_{2,0})x_{2,0}} \sum_{m_1=0}^{n_1-1} \sum_{m_2=0}^{n_2-1} e^{i\delta u_1\delta x_{1,j_1m_1} + i\delta u_2\delta x_{2,j_2m_2}} e^{iu_1x_{1,m_1} + iu_2x_{2,m_2}} \bar{a}_{m_1m_2}.$$ 

The definitions of the one-variable (inverse) Fourier transforms in (21)-(22) follow from straightforward reduction of the results in Appendix A to the one-variable case. These are properly proved in Bochner and Chandrasekharan ((1949), Theorems 1, 8-9, 2).
and

\[
\hat{a}_{m_1m_2} = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} e^{-iu_{1,j_1}x_1 - iu_{2,j_2}x_2} \bar{b}_{j_1j_2}
\]

\[
= e^{-iu_{1,0}(x_1-x_{1,0}) - iu_{2,0}(x_2-x_{2,0})} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} e^{-i\delta u_1 \delta x_1 j_1 m_1 - i\delta u_2 \delta x_2 j_2 m_2} e^{-iu_{1,j_1}x_1 - iu_{2,j_2}x_2} \bar{b}_{j_1j_2}.
\]

We denote \( \bar{B} = D(\bar{A}, x_1, x_2)(u_1, u_2) \) and \( \bar{A} = D^{-1}(\bar{B}, u_1, u_2)(x_1, x_2) \).

Hence, upon imposing the Nyquist restrictions \( \delta u_1 \delta x_1 = \frac{2\pi}{n_1} \) and \( \delta u_2 \delta x_2 = \frac{2\pi}{n_2} \), we have

\[
\bar{B} = D(\bar{A}, x_1, x_2)(u_1, u_2) = e^{iu_{1,0}x_1} e^{iu_{2,0}x_2} \cdot i fft2\left(e^{iu_{1,0}x_1} e^{iu_{2,0}x_2} \cdot \bar{A}\right)
\]

and

\[
\bar{A} = D^{-1}(\bar{B}, u_1, u_2)(x_1, x_2) = e^{-iu_{1,0}(x_1-x_{1,0})} e^{-iu_{2,0}(x_2-x_{2,0})} \cdot fft2\left(e^{-iu_{1,0}x_1} e^{-iu_{2,0}x_2} \cdot \bar{B}\right).
\]

Based on the previous definitions, the convolution algorithm for the valuation of the convertible bond works as follows.

1. In order to compute numerically the Fourier transforms \( \mathcal{F}(\tilde{H}_{k-}) \) and \( \mathcal{F}^{-1}(\mathcal{F}(\tilde{H}_{k-}) \phi^*) \) in (19) and (20), we first need to determine the truncated and discrete uniformly spaced grids \( (y, y_r) \) and \( (u, u_r) \), which must obey to the Nyquist constraints. Although proper bounds for the induced truncation and discretization errors are not explicitly available, it is easy to test possible choices for the upper and lower limits of the grid ranges, while trading off the grid spacing for a given number of grid points. We also need to ensure that the surface \( |\phi^*| \) is negligible outside the truncated \( uu_r\)-plane.

2. Suppose that the values of \( \tilde{H}_{k-} \) are approximated along the pre-specified uniform grid \( (y, y_r) \) and the values of \( \mathcal{F}(\tilde{H}_{k-}) \) and \( \phi^* \) along the grid \( (u, u_r) \); assume, further, that these discrete values of \( \tilde{H}_{k-} \) and \( \phi^* \) over the corresponding grids are given by the matrices \( \tilde{H}_{k-} \)
and $\phi^*$. Then, we evaluate $N\delta y N_r \delta y_r \mathcal{D}(\tilde{H}_{k-1} \cdot w, y, y_r)(u, u_r)$ as the discrete approximation of $\mathcal{F}(\tilde{H}_{k-1})$, where $N$ refers to the number of grid points along $y$ and $u$, $N_r$ is the number of grid points along $y_r$ and $u_r$ and

$$w = \{w_{j,j_r} = w_j w_{j_r}\}_{j=0,j_r=0}^{N-1,N_r-1}; \ w_j = \begin{cases} \frac{1}{2}, & j = 0, N \\ 1, & \text{otherwise} \end{cases}, \ w_{j_r} = \begin{cases} \frac{1}{2}, & j_r = 0, N_r \\ 1, & \text{otherwise} \end{cases}$$

are the trapezium rule weightings. For some uniform grid $(x, x_r)$, the approximate inverse Fourier transform $\hat{H}_{k-1}^{-1} = \mathcal{F}^{-1}(\mathcal{F}(\tilde{H}_{k-1})\phi^*)$ is provided by

$$\hat{H}_{k-1}^{-1} = \frac{\delta u \delta u_r}{(2\pi)^2} \mathcal{D}^{-1}(N\delta y N_r \delta y_r \mathcal{D}(\tilde{H}_{k-1} \cdot w, y, y_r)(u, u_r) \cdot \phi^*, u, u_r)(x, x_r)$$

where the last equality holds by the linearity of the $\mathcal{D}^{-1}$ operator and the Nyquist restrictions.

(3) We approximate the values $\hat{H}_{k-1}$ at $g_{k-1} (y, y_r) \subseteq x, g_{r,k-1} (y_r) \subseteq x_r$ by fitting a surface to the nodes $(x, x_r, \hat{H}_{k-1})$ using cubic interpolation, whereas for $g_{k-1} (y, y_r) \not\subseteq x$ we extrapolate linearly in $e^x$. For the $N \times N_r$ matrix $Y_r$ whose rows coincide with the row vector $y_r$, we compute

$$\mathcal{H}_{k-1} = P_{t_{k-1}}(t_k) \cdot \hat{H}_{k-1} \left( g_{k-1} (y, y_r), g_{r,k-1} (y_r) \right)$$

\[= \exp(A_{t_{k-1}}(t_k) - B_{t_{k-1}}(t_k) Y_r) \cdot \hat{H}_{k-1} \left( g_{k-1} (y, y_r), g_{r,k-1} (y_r) \right)\]

on $(y, y_r)$.

(4) We calculate $\tilde{H}_{k-1}$ in accordance with the corresponding payoff function (8)-(11) and we are automatically back to step 2. Additionally, upon a call announcement date, the approximate values for $\tilde{K}_{r_c}$, as by (15)-(16), are obtained after following a straightforward
reduction of steps 1-3 to the case of the one-variable (inverse) Fourier transforms (21)-(22), and making use of the MATLAB functions ifft and fft, with relevant vectors as their inputs and outputs rather than matrices. Furthermore, in step 3, $\hat{K}_c$ is computed at $h_c(y, y_r)$ only twice (for $\tau^c \neq t_{j-}$ and $\tau^c = t_{j-}$) for any number of time steps. The recursive scheme for $H$ is operated until we reach $H_0$.

We note the following. As a possible alternative to the standard DFTs developed in (23) and their computation via the (inverse) FFT routine, one may define chirp z-transforms and compute them as suggested in Černý and Kyriakou (2010). The chirp z-transforms allow us to construct the spatial and Fourier meshes independently (the Nyquist restrictions are abandoned), although at a higher computational cost (three standard DFT evaluations per chirp z-transform evaluation). For this reason, unless we discard the CB’s call feature, which entails systematic monitoring of the firm value process and therefore raises substantially the required number of time steps, the chirp z-transforms are recommended only upon excluding stochastic interest rates. As far as the choice of the integration weights is concerned, preliminary testing of our procedure has shown that the convergence of the scheme is significantly affected by the choice of $w$. In particular, with high-order Newton-Côtes rules, like the Simpson’s rule, we have encountered non-smooth convergence issues, as opposed to the low-order trapezium rule (see also Lord et al. (2008), Section 4.2).

V Numerical results

A Black-Scholes-Merton model

In order to illustrate the performance of our algorithm, we choose as our benchmark the prices obtained from the exact analytical formula of Ingersoll (1977a) for continuously callable CBs in the Black-Scholes-Merton economy under the assumption of constant interest rates. For testing purposes, we focus on CBs which mature in 5 years (typical) and 2 years. In Table 1...
present our results for finite sampling frequency and infinite sampling frequency, alongside the prices obtained from the closed form solution. Table I reports both the cases of constant and time-dependent call prices.

Upon constant interest rates, we can achieve results, subject to finite monitoring frequency, which are precise to 7 decimal places. Furthermore, precision to 4 decimal places is attainable in 1.1 sec when $T = 2$, $n = 500$ (i.e., daily sampling), and in 2.2 sec when $T = 5$, $n = 1250$ (daily sampling). By regular convergence of our scheme, we can additionally extrapolate the discretely monitored CB prices to approximate the price of an otherwise equivalent, continuously monitored CB (infinite sampling). We then generate results subject to less than 0.0005% error. For given time to maturity, this precision can be improved if we raise the sampling frequency. We reach the same conclusions on the assumptions of constant call price and call price as function of time. When stochastic interest rates are assumed, the results currently produced are accurate to the fourth decimal place and the third decimal place for $T = 2$ and 5, respectively. Although higher-level accuracy is possible by the numerical scheme proposed, finer 2-D grids cannot be accommodated easily due to technological insufficiency (at least for PC users). The case of stochastic interest rates is revisited in Section V.C.

Here, our recursion proves competitive with standard numerical techniques, given also the number of risk factors they can flexibly accommodate. In particular, Ammann et al. (2008) simulate callable CB prices ($T = 2$, daily sampling) in a two-factor setting with stochastic interest rates. Monte Carlo price estimates are reported up to the second decimal place, subject to standard error of order $10^{-1}$ (stochastic interest rates) and $10^{-2}$ (constant interest rates). Standard errors of variable order $10^{-1}$-$10^{-2}$ are also common in the two-factor joint simulation-regression application in Lvov et al. (2004) (subject to 16 exercise times per year). In the PDE/I context with two-factors, Zvan et al. (2001) obtain monotone convergence in the number of grid and time points, which they attribute, nevertheless, to the conversion and call boundary

\footnote{Hereafter, all CPU times reported are for MATLAB R2007b on an Intel Core 2 Duo processor T5500, 1.66GHz, 2GB of RAM.}

26
Callable CB prices in the Black-Scholes-Merton model

Callable CB specification: $F = 40$, $C = D = 0$, $m = 1$, $\gamma = 0.2$, $\vartheta = 0$, $s_c = 0$. Firm value parameters: $V_0 = 100$, $\sigma = 0.25$. Constant interest rate: $r = 0.04$. The error is expressed as a percentage of the exact price obtained using the result by Ingersoll (1977a).

<table>
<thead>
<tr>
<th>$K_t$</th>
<th>$T$</th>
<th>$n$</th>
<th>Ballotta &amp; Kyriakou: constant interest rates</th>
<th>Ingersoll: constant interest rates</th>
<th>error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>2</td>
<td>250</td>
<td>36.9477708</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>500</td>
<td>36.9490640</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\infty$</td>
<td>36.95217</td>
<td>36.9522338</td>
<td>0.00017</td>
</tr>
<tr>
<td>5</td>
<td>500</td>
<td>33.1045746</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1250</td>
<td>33.1191553</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td>33.14434</td>
<td>33.1444004</td>
<td>0.00018</td>
<td></td>
</tr>
<tr>
<td>$40e^{-0.02(T-t)}$</td>
<td>2</td>
<td>250</td>
<td>36.9306700</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>500</td>
<td>36.9313884</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\infty$</td>
<td>36.93311</td>
<td>36.9331542</td>
<td>0.00012</td>
</tr>
<tr>
<td>5</td>
<td>500</td>
<td>32.8570091</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1250</td>
<td>32.8659022</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td>32.88121</td>
<td>32.8813609</td>
<td>0.00046</td>
<td></td>
</tr>
</tbody>
</table>
conditions forcing the solution to be closely linear over large parts of the spatial domain. They report callable CB prices \((T = 10, n = 320)\) with precision up to three decimal places. Barone-Adesi et al. (2003) and Bermúdez and Webber (2004) employ a joint characteristics-finite elements scheme to price callable CBs \((T = 5, n = 400)\), which converges at first order in the number of grid and time steps. Although they achieve accuracy up to 3 decimal places, their PDI method suffers by increasing dimensionality when random jumps are included into the firm value dynamics. For this reason, they resort to the simplifying assumption of a single jump of fixed size, in order to maintain the 2-D structure of their PDI.

\section*{B \ Jump diffusion setup}

In this section, we examine the impact of including jumps in the original Gaussian log-return diffusion and how variations of the jump intensity \(\lambda\), mean \(\mu_L\) and variance \(\sigma_L\) of the jump size affect the callable CB prices. To this aim, we ignore for convenience and without loss of generality the case of stochastic interest rates, due to their independence from the jump component, and calibrate the DEJD and MJD risk-neutral models to match mean and variance of the log-return process as well as \(\lambda\), \(\mu_L\) and \(\sigma_L\). The base values for these quantities are consistent with the assumptions of Dao and Jeanblanc (2006). The fitted parameters and moments, including the resulting skewness coefficient and the excess kurtosis index, are summarized in Table 2 (the cumulants of the log-return process are obtained in Appendix D).

The performance in terms of accuracy and computational time of the FFT-based algorithm has already been tested in the \(\text{Lévy} \) setting (including the tempered stable and normal inverse Gaussian models) by Černý and Kyriakou (2010) for the pricing of other derivative contracts, such as discretely sampled Asian options. In the case of the MJD and DEJD setups considered in this paper, the numerical method shows similar robustness across different levels of moneyness of the convertible bond and model parameter values.
Table 2: Calibrated model parameters

The parameters $r = 0.04$, $\sigma = 0.2$ remain fixed in all cases. We set $I_t = \ln(V_t/V_0)$; this process has mean $\mathbb{E}(I_t)$ and variance $\mathbb{V}ar(I_t)$, whilst $s(I_t)$ and $\kappa(I_t)$ denote the skewness coefficient and excess kurtosis index, respectively. These quantities are calculated according to the results shown in Appendix D.

<table>
<thead>
<tr>
<th>case</th>
<th>MJD &amp; DEJD</th>
<th>MJD</th>
<th>DEJD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda$</td>
<td>$\mu_L$</td>
<td>$\sigma_L$</td>
</tr>
<tr>
<td>Base</td>
<td>3</td>
<td>-0.0150</td>
<td>0.0357</td>
</tr>
<tr>
<td>$\lambda_I$</td>
<td>5</td>
<td>-0.0150</td>
<td>0.0357</td>
</tr>
<tr>
<td>$\lambda_{II}$</td>
<td>1</td>
<td>-0.0150</td>
<td>0.0357</td>
</tr>
<tr>
<td>$\mu_{L, I}$</td>
<td>3</td>
<td>-0.0300</td>
<td>0.0357</td>
</tr>
<tr>
<td>$\mu_{L, II}$</td>
<td>3</td>
<td>-0.0075</td>
<td>0.0357</td>
</tr>
<tr>
<td>$\sigma_{L, I}$</td>
<td>3</td>
<td>-0.0150</td>
<td>0.0714</td>
</tr>
<tr>
<td>$\sigma_{L, II}$</td>
<td>3</td>
<td>-0.0150</td>
<td>0.0179</td>
</tr>
</tbody>
</table>

In Table 3, we study the average price deviation\footnote{Actual prices used in the computations are available from the authors upon request.} between the two paradigms, as function of the parameter values $\lambda$, $\mu_L$, $\sigma_L$ and the moneyness of the CB. Moneyness is calculated as the ratio between the conversion and investment values, where the latter is defined as the hypothetical bond value in the absence of the conversion option and the credit risk.

Several comments are in order. In all cases, the MJD model prices are in excess of the prices generated by the DEJD model. This is due to the constantly stronger negative skewness and leptokurtosis of the DEJD distribution, which together guarantee higher and lower likelihoods of default and call, respectively. The reduction in the value caused by the default effect is strong enough to overshadow the raise in the CB value caused by the call effect. The marked asymmetry of the DEJD distribution and the corresponding excess kurtosis are due to the fact that the event of a downward jump is more likely under every parameter combination considered here. In fact, as a result of imposing the same rate of arrival, $\lambda$, the same mean and variance for both the jump sizes, $L$, and the log-returns across the two models, the parameter $p$, i.e., the probability assigned to an upward jump, is always less than 0.5 regardless of the mean.
Table 3: MJD versus DEJD, non-leptokurtic (Gaussian) versus leptokurtic distribution

Estimated average price difference as function of $\lambda$, $\mu_Z$, $\sigma_Z$ and CB moneyness. As benchmark for the leptokurtic case, we choose the DEJD model as this represents the stronger departure from the Gaussian case. Callable CB specification: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $C = D = 0$, $m = 1$, $\gamma = 0.2$, $\vartheta = 0$, $s^e = 0$. Prices (with accuracy up to 5 decimal places) computed for $\ln V_0 = [\ln 100, \ln (K/\gamma)]$ split into 17,150 equidistant points. Moneyness ranges from 0.6 to 1.5. Prices differences obtained and averaged piecewise for moneyness regions $[0.6, 1)$, $[1, 1.2)$, $[1.2, 1.4)$ and $[1.4, 1.5]$. For deep in-the-money CBs (top moneyness slice), the average price difference tends practically to zero level due to the firm’s highly likely call.

<table>
<thead>
<tr>
<th>case</th>
<th>MJD - DEJD</th>
<th>non-leptokurtic - leptokurtic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>moneyness</td>
<td>moneyness</td>
</tr>
<tr>
<td></td>
<td>0.6-1</td>
<td>1-1.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.2-1.4</td>
</tr>
<tr>
<td>Base</td>
<td>0.006</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.016</td>
</tr>
<tr>
<td>$\lambda_I$</td>
<td>0.009</td>
<td>0.006</td>
</tr>
<tr>
<td></td>
<td>0.002</td>
<td>0.017</td>
</tr>
<tr>
<td>$\lambda_{II}$</td>
<td>0.010</td>
<td>0.001</td>
</tr>
<tr>
<td>$\mu_{L, I}$</td>
<td>0.110</td>
<td>0.021</td>
</tr>
<tr>
<td>$\mu_{L, II}$</td>
<td>0.003</td>
<td>0.001</td>
</tr>
<tr>
<td>$\sigma_{L, I}$</td>
<td>0.023</td>
<td>0.050</td>
</tr>
<tr>
<td>$\sigma_{L, II}$</td>
<td>0.001</td>
<td>0.003</td>
</tr>
</tbody>
</table>

30
size of the up/downward jump (controlled by $\eta_1$, $\eta_2$ respectively). The observed skewness and kurtosis also explain the higher prices generated by the Gaussian model, as this underestimates the probability of default.

Further, Table 2 shows that, in the $\lambda_{II}$, $\mu_{L,II}$, $\sigma_{L,II}$ cases, the effect of the jump component is negligible; Table 3 confirms in fact that the prices originated by the two jump diffusion processes and the Gaussian model coincide to penny accuracy. On the contrary, in the $\lambda_{I}$, $\mu_{L,I}$, $\sigma_{L,1}$ cases, when the presence of the jump part is more significant, the MJD versus the DEJD price deviation reaches up to five pence, whilst the non-leptokurtic versus leptokurtic deviation can be up to 15 pence. Changes in $\sigma_L$ appear to have the most noticeable impact on the price discrepancy among the three parameter-type modifications we consider here, due to the higher impact that this parameter has on the overall skewness and kurtosis of the log-returns.

Moreover, in all cases, for deep in-the-money CBs, all the models’ prices converge to the call price since the CB is then forced-by-call converted. For $\mu_L$ which is well below zero, the MJD versus DEJD price difference is observed to peak from an early stage, when the CB is close to the money, while for $\mu_L$ closer to zero, the peak delays until the CB is in the money. This behaviour is attributed to the different level of skewness and kurtosis originated by the two different combinations of parameters associated to $\mu_{L,1}$ and $\mu_{L,II}$, and therefore the different impact of the default effect and the call effect, as previously discussed. Similar pattern, with higher-level peak though, is spotted for the two cases of $\sigma_L$ considered here.

C Effects of discrete coupon and dividend payments

We explore the consequences of adding discrete coupons and dividends into the valuation framework. In Tables 4 and 5, we report prices for 5-year callable CBs on a daily sampling basis ($n = 1250$), subject to both constant and stochastic interest rates. In the case of stochastic interest rate, we select $r_0 = \mu_r = 0.04$, $\rho = 0.2$ and $\kappa = 0.858$, $\sigma_r = 0.047$, as estimated by Aït-Sahalia (1996) for the Vašíček model, whereas we set $r = r_0$ for the constant interest rate.
rates assumption. We employ the base parameter set as in Table 2, and, additionally, assume \( V_0 = 100,\) \( F = 40,\) \( K = 50,\) \( C = 1 \) (payable at middle and end of the year), \( D = 2 \) (payable in three and nine months time), \( m = 1,\) \( \gamma = 0.2,\) \( s^c = 0,\) \( \vartheta = 0.\) We adopt here the scaled constant \( F,\) \( C \) and \( D \) values, as in the example of Brennan and Schwartz ((1977), Section V).

In the case of stochastic interest rates, the precision of the reported numbers is up to the third decimal place. This is feasible by employing Richardson extrapolation, due to the smooth linear convergence of the scheme in the number of grid points. When the interest rate is constant, we acquire higher CPU power and produce results precise to the fifth decimal place, although higher accuracy (up to 7 decimal places) is possible via Richardson extrapolation. It becomes obvious from Tables 4 and 5 that the CPU timings rise from the constant to the stochastic interest rates setup, and from the simply callable CB to the coupon-bearing one and to another one with associated dividend-paying stock. In fact, the increase originated by the introduction of stochastic interest rates is due to the change from 1-D to 2-D Fourier transforms, whereas the additional computational times required by a callable CB with coupons and with both coupons and dividends are due to the need to approximate the CB values at the relevant time points on three different costly 2-D grids (see step 3 in Section IV[A]). In any case, we do not exceed the typical 6700 sec Fortran execution time, independent of the contract specification, of the PDI implementation reported in Bermúdez and Webber (2004).

Few comments are in order. Adding coupons in the bond indenture raises substantially the payoff to the investors and, consequently, the CB value. At the same time, the firm value and, consequently, the chances for a call reduce, increasing in this way the value of the CB, whilst the default event becomes more likely, negatively affecting the CB value. Nevertheless, the first two effects beat the third one, justifying the overall increase in the CB value observed. Moreover, for a dividend-paying common stock, a decline in the contract’s price is noticed. This occurs because the dividends are not payable to the CB holders pre-conversion and, at the same time, they affect the rate at which the firm value appreciates, boosting, in this way, the chances of future default.
Table 4: Constant interest rates: comparison between MJD and DEJD prices for different CB specifications. The CPU times reported correspond to accuracy up to 5, 4, 3 decimal places.

<table>
<thead>
<tr>
<th>CB specification</th>
<th>model MJD</th>
<th>price differences ×10⁻⁴</th>
<th>CPU (sec) prec ±10⁻⁵</th>
<th>prec ±10⁻⁴</th>
<th>prec ±10⁻³</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>33.40333</td>
<td>53</td>
<td>220</td>
<td>2.2</td>
<td>1.1</td>
</tr>
<tr>
<td>call, coupons</td>
<td>41.91545</td>
<td>45</td>
<td>250</td>
<td>2.4</td>
<td>1.2</td>
</tr>
<tr>
<td>call, coupons, dividends</td>
<td>40.76062</td>
<td>48</td>
<td>265</td>
<td>4.8</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Table 5: Stochastic interest rates: comparison between MJD and DEJD prices for different CB specifications. The CPU times reported correspond to accuracy up to 3, 2 decimal places.

<table>
<thead>
<tr>
<th>CB specification</th>
<th>model MJD</th>
<th>price differences ×10⁻⁴</th>
<th>CPU (sec) prec ±10⁻³</th>
<th>prec ±10⁻²</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>33.655</td>
<td>50</td>
<td>1520</td>
<td>410</td>
</tr>
<tr>
<td>call, coupons</td>
<td>42.089</td>
<td>40</td>
<td>3530</td>
<td>680</td>
</tr>
<tr>
<td>call, coupons, dividends</td>
<td>40.702</td>
<td>50</td>
<td>5740</td>
<td>930</td>
</tr>
</tbody>
</table>

Furthermore, the discrepancy between the MJD and the DEJD model prices remains positive and smaller than the average computed for the out-of-the-money CBs, reported in Table 3 for the base case parameters. This price difference reduces in the case of a coupon-bearing CB, since the coupons have a primary positive upshot on the value of the bond, reducing the impact of the stronger negative skewness and fatter tails of the DEJD, as compared to the MJD.

D Effects of call policy

In order to investigate the consequences of the adopted call strategy, we assume a typical call notice period of a month ($s_c = 1/12$) and safety premium $\vartheta = 0.2$, and examine how changes from these initial values affect the model prices produced under the base case and the $\sigma_{L,1}$ parameter set in Table 2. We generate prices for callable CBs with 5 years to maturity subject
Table 6: Base parameter set, constant interest rates: callable CB prices for varying call specification \((s^c, \vartheta)\). Precision up to 5 decimal places.

<table>
<thead>
<tr>
<th>((s^c, \vartheta))</th>
<th>model</th>
<th>(s^c = 1/12) vs (1/24)</th>
<th>(\vartheta = 0.20) vs (0.25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1/12, 0.20)</td>
<td>33.20939 33.20351</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/12, 0.25)</td>
<td>33.14899 33.14321</td>
<td>0.06040 0.06030</td>
<td></td>
</tr>
<tr>
<td>(1/24, 0.20)</td>
<td>33.20935 33.20345</td>
<td>0.00004 0.00006</td>
<td></td>
</tr>
<tr>
<td>(1/24, 0.25)</td>
<td>33.14899 33.14319</td>
<td>&lt; (10^{-5}) 0.00002</td>
<td></td>
</tr>
</tbody>
</table>

Table 7: \(\sigma_{L,I}\) parameter set, constant interest rates: callable CB prices for varying call specification \((s^c, \vartheta)\). Precision up to 5 decimal places.

<table>
<thead>
<tr>
<th>((s^c, \vartheta))</th>
<th>model</th>
<th>(s^c = 1/12) vs (1/24)</th>
<th>(\vartheta = 0.20) vs (0.25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1/12, 0.20)</td>
<td>33.40978 33.40891</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/12, 0.25)</td>
<td>33.33957 33.33896</td>
<td>0.07021 0.06995</td>
<td></td>
</tr>
<tr>
<td>(1/24, 0.20)</td>
<td>33.40940 33.40846</td>
<td>0.00038 0.00045</td>
<td></td>
</tr>
<tr>
<td>(1/24, 0.25)</td>
<td>33.33946 33.33877</td>
<td>0.00011 0.00019 0.06994 0.06969</td>
<td></td>
</tr>
</tbody>
</table>

We ignore discrete coupons and dividends in this section. For the interest rates model we assume the same parameters as in Section V.C. The precision of the reported numbers is 5 decimal places (achieved in 270 sec) and 3 decimal places (achieved in 1570 sec) for constant and stochastic interest respectively.

In general, increasing \(\vartheta\) and/or \(s^c\) raises the chances for a successful forced-by-call conversion at the end of the call notice period and, hence, reduces the CB value. According to the indications in Tables 6-8, the call notice has a minor effect on the CB values. This is because of the short call notice lengths (30 days, and 15 days) as compared to the longer time to maturity of the CB issue (5 years). Furthermore, when we change from the base case to the \(\sigma_{L,I}\) parameters set, which originates strongest variance, skewness and kurtosis features (see Table 9), we reached the same conclusion for \(T = 2\).
Table 8: Left panel: base parameter set; right panel: $\sigma_{L,1}$ parameter set, stochastic interest rates: callable CB prices for varying call specification $(s^c, \vartheta)$. Precision up to 3 decimal places.

<table>
<thead>
<tr>
<th>$(s^c, \vartheta)$</th>
<th>model MJD DEJD</th>
<th>$\vartheta = 0.20$ vs $0.25$ MJD DEJD</th>
<th>model MJD DEJD</th>
<th>$\vartheta = 0.20$ vs $0.25$ MJD DEJD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1/12, 0.20)$</td>
<td>33.451 33.445</td>
<td>0.066 0.065</td>
<td>33.633 33.634</td>
<td>0.073 0.073</td>
</tr>
<tr>
<td>$(1/12, 0.25)$</td>
<td>33.385 33.380</td>
<td></td>
<td>33.560 33.561</td>
<td></td>
</tr>
<tr>
<td>$(1/24, 0.20)$</td>
<td>33.451 33.445</td>
<td></td>
<td>33.633 33.634</td>
<td></td>
</tr>
<tr>
<td>$(1/24, 0.25)$</td>
<td>33.385 33.380</td>
<td>0.066 0.065</td>
<td>33.560 33.561</td>
<td></td>
</tr>
</tbody>
</table>

we observe at best an increase in the MJD and DEJD differences across the two $s^c$ choices by a factor of 10, although the actual deviation does not exceed $4 \times 10^{-4}$.

On the contrary, the choice of the safety premium, which relates to the firm’s decision on the date of the call announcement, appears to be a primary factor driving the CB values. In the case of the $\sigma_{L,1}$ parameters combination, for given $\vartheta$, the prices increase substantially reflecting the higher possibility of a failed call, which implies that the issuing firm should raise the safety premium, $\vartheta$, as protection against a potentially failed call. The MJD and DEJD differences across the two $\vartheta$ values under consideration are about $6 \times 10^{-2}$ and $6.5 \times 10^{-2}$ for the base parameters set, and $7 \times 10^{-2}$ and $7.3 \times 10^{-2}$ for the $\sigma_{L,1}$ parameters set, in the case of constant and stochastic interest rates respectively. The effect caused by $\vartheta$ when the firm value log-return is more volatile, is more pronounced due to the increase in the skewness and kurtosis, similarly to what observed in Section V.B. The increase in $\vartheta$, in fact, reduces the probability of the call price to be paid at the end of the notice period.

VI Conclusions

The main contribution of this paper consists in the development and implementation of a numerical pricing scheme for convertible bonds based on Fourier transform techniques. The proposed method is shown to be efficient and accurate, and to flexibly accommodate a number
of contract-design features such as callability provisions, dividends and coupon payments. The procedure has also been shown capable of coping with up to four risk factors, allowing a market setup based on a jump diffusion-driven underlying asset for the CB, and stochastic interest rates.

The proposed numerical scheme requires the knowledge of the characteristic function of the bivariate log-firm value-interest rate process, which has been derived in this paper using the change of numéraire technique coupled with results on affine structure models.

The pricing methodology presented in this paper has been tested using several parameter sets of the market model. We first benchmarked the Fourier transform algorithm against the closed form solution obtained by Ingersoll (1977a) for the case of a continuously callable CB in the Black-Scholes framework with constant interest rate. The numerical results showed the accuracy of the algorithm up to 7 decimal places, for varying monitoring frequency. The analysis has been then extended first to the more general jump diffusion setting with constant interest rates, then to the case of a stochastic term structure.

As a jump-diffusion market setup for pricing CBs is new in the literature, we also used the proposed algorithm to analyze the behaviour of the contract price under this more complex representation of the firm value. The main results of the numerical analysis show that the jump diffusion setup originates lower values of the CB when compared to the classical Black-Scholes framework. This is essentially due to the higher probability of default generated by the inclusion of market shocks in the model, i.e., by the negatively skewed and leptokurtic distribution of the log-firm value resulting from the jump diffusion setup.

The analysis offered in this paper does not provide bounds for the truncation and discretization errors; it is however possible to test different choices of the upper and lower limits for the range of grid values and find ones which diminish the discretization and truncation effects, while preserving the smooth convergence of the scheme. Another issue that is not dealt with in this paper is the calibration of the market model. As discussed in Section II.B, in fact, we adopt a structural approach to credit risk in the same spirit as Merton (1974). However, the fact
that the firm value is not directly observable in the market, leaves open the problem of model calibration. In Section II.C we discuss few possible solutions to this issue, the implementation and testing of which is left for future research.

Finally, we note that the CB valuation scheme we propose here is general enough to accommodate, for example, a stock-based setting, as opposed to the current firm value-based model, with jumps and stochastic interest rates, for suitably modified CB payoff functions (8)-(11) (see, for instance, Goldman Sachs (1994) and Barone-Adesi et al. (2003)). Also, if necessary, the put provision can be flexibly incorporated into the payoff functional (see, for example, Goldman Sachs (1994)). Apart from the computation of the CB prices, our method can be extended to the computation of the price sensitivities (Greeks). Moreover, the extensive CB pricing scheme we suggest here can be easily reduced and specialized to the pricing of simpler exotic derivatives and, in fact, extend the works of Lord et al. (2008) on the pricing of Bermudan/American vanilla and discretely monitored barrier options to the case of stochastic interest rates.

Appendix

A Fourier transforms in $L^1$ (two variables)

Define the Fourier transform of a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, where $f \in L^1$, by

$$
(A.1) \quad \mathcal{F}(f)(u) = \int_{\mathbb{R}^2} e^{iu \cdot s} f(s) \, ds; \quad u \in \mathbb{R}^2.
$$

Then for every $u \in \mathbb{R}^2$, $\mathcal{F}(f)(u)$ is bounded and $\lim_{u \cdot u^T \to \infty} \mathcal{F}(f)(u) = 0$ (Bochner and Chandrasekharan (1949), Theorem 31). Bochner and Chandrasekharan ((1949), Theorems 37, 38, 39) show that the Fourier transform of a Lebesgue integrable function is Gauss-summable and on this argument they derive the inversion formula for (A.1). In particular, if $f \in L^1$ and
\( F(f) \in L^1 \) in \( \mathbb{R}^2 \), or alternatively \( f \in L^1 \) is bounded and \( F(f) \geq 0 \), then

\[
f(s) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-iu \cdot s} F(f)(u) \, du
\]

for almost all \( s \in \mathbb{R}^2 \). It is sufficient to add continuity to \( f(s) \) for the inversion formula to hold everywhere.

Furthermore, if functions \( f_1, f_2 \in L^1 \), then their convolution which is defined as

\[
(f_1 \ast f_2)(s) \doteq \int_{\mathbb{R}^2} f_1(s') f_2(s - s') \, ds' = \int_{\mathbb{R}^2} f_2(s') f_1(s - s') \, ds',
\]

exists for almost all \( s \in \mathbb{R}^2 \), belongs to \( L^1 \), and

\[
F(f_1 \ast f_2) = F(f_1)F(f_2)
\]

(Bochner and Chandrasekharan (1949), Theorem 33).

**B Equivalent martingale measure changes**

Knowledge about the distribution of the log-firm value under the forward-risk-adjusted probability measure is required for the computation of \( \tilde{K}_c \). We deal with changes of the underlying distribution, which follow changes from the risk-neutral pair to a different numéraire pair, using the results of Geman et al. (1995).

**Proposition 2** For the pure-discount bond price \( P_t(v) \) in (2), there exists a martingale measure \( \mathbb{P}^* \) defined by its Radon-Nikodým derivative with respect to \( \mathbb{P} \)

\[
\gamma_t^* = \left. \frac{d\mathbb{P}^*}{d\mathbb{P}} \right|_{F_t} = \exp \left( -\rho^2 \int_0^t m_s^2(v) \, ds - (1 - \rho^2) \int_0^t m_s^2(v) \, ds \right.
\]

\[
+ \rho \int_0^t m_s(v) \, dW_s + \sqrt{1 - \rho^2} \int_0^t m_s(v) \, d\tilde{W}_s \bigg), \quad t \in [0, v],
\]
where $W$ and $\tilde{W}$ are independent $\mathbb{R}$-valued $\mathbb{F}$-adapted $\mathbb{P}$-standard Brownian motions. We define $\mathbb{P}^*$ to be the ‘forward-risk-adjusted’ probability measure, which is associated to the numéraire $P_t(v)$.

**Proof.** The pure-discount bond price $P_t(v)$ satisfies the definition of a numéraire, as in Geman et al. (1995). Then, based on Geman et al. ((1995), Theorem 1), we construct the Radon-Nikodým derivative

$$
\gamma^*_t \equiv \frac{d\mathbb{P}^*}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \frac{P_t(v)}{\beta_t P_0(v)},
$$

where $\beta_t = \exp \left( \int_0^t r_s ds \right)$. From (2), we have that

$$
\gamma^*_t = \exp \left( - \int_0^t \frac{m_s^2(v)}{2} ds + \int_0^t m_s(v) dW_{r,s} \right)
$$

$$
= \exp \left( -\rho^2 \int_0^t \frac{m_s^2(v)}{2} ds - (1 - \rho^2) \int_0^t \frac{m_s^2(v)}{2} ds + \rho \int_0^t m_s(v) dW_s + \sqrt{1 - \rho^2} \int_0^t m_s(v) d\tilde{W}_s \right),
$$

which follows in virtue of the decomposition $W_r = \rho W + \sqrt{1 - \rho^2} \tilde{W}$, for independent Brownian motions $W$ and $\tilde{W}$. ■

Based on Proposition 2, we conclude, in virtue of the Girsanov theorem, that $W$ and $W_r$ retain their semimartingale property and decompose, after the martingale measure change, to

(A.2) \hspace{1cm} W_t = W^*_t + \int_0^t \rho m_s(v) ds,

(A.3) \hspace{1cm} W_{r,t} = W^*_{r,t} + \int_0^t m_s(v) ds,

where $W^*$ and $W^*_r$ are correlated $\mathbb{P}^*$-standard Brownian motions with constant correlation $\rho \in (-1, 1)$.
C Characterization of the bivariate log-firm value-interest rate process under the $\mathbb{P}^*$ equivalent martingale measure

In the subsequent derivation, for compactness we denote $E(\cdot | \mathcal{F}_t) = E_t$ under any applicable probability measure.

Following Geman et al. (1995), Corollary 2 and the change to the $\mathbb{P}^*$ measure as described in Proposition 2, we have that

\[(A.4) \quad E^*_s (\exp (iv_1 Y_t + iv_2 r_t)) = \frac{E_s \left( \exp \left( - \int_s^t r_u du + iv_1 Y_t + iv_2 r_t \right) \right)}{P_s (t)}.
\]

Given the independence of the jump part of the jump diffusion process from the diffusion part and the short rate, we restate the right-hand side of equation (A.4) as

\[(A.5) \quad \frac{1}{P_s (t)} E^*_s \left( \exp \left( - \int_s^t r_u du + iv_1 \left( Y_s + \int_s^t \left( r_u - \frac{\sigma^2}{2} - \lambda (\phi_L (-i) - 1) \right) du \right. \right. \right.

\[ \left. \left. + \sigma \int_s^t dW_u \right) + iv_2 r_t \right) \right) \right) E_s \left( \exp \left( iv_1 \int_{\mathbb{R}} l (N_s (dl) - N_s (dl)) \right) \right) .
\]

Then, equation (A.5) is equivalent to

\[(A.6) \quad \frac{1}{P_s (s + \tau)} \exp \left( Y_s + \frac{\sigma^2}{2} \frac{iv_1}{iv_1 - 1} \tau \right) E_s \left( \exp \left( (iv_1 - 1) (Y_s \right. \right.

\[ \left. + \int_s^{s+\tau} \left( r_u - \frac{1}{2} \left( \sigma \frac{iv_1}{iv_1 - 1} \right)^2 \right) du + \sigma \frac{iv_1}{iv_1 - 1} \int_s^{s+\tau} dW_u \right) + iv_2 r_{s+\tau} \right) \right) \times \exp \left( -i \lambda (\phi_L (-i) - 1) v_1 \tau \right) E_s \left( \exp \left( iv_1 \int_{\mathbb{R}} l (N_{s+\tau} (dl) - N_s (dl)) \right) \right) ,
\]

where $\tau = t - s$. Next, we define

\[\tilde{\sigma} \doteq \frac{\sigma v_1}{v_1},\]

\[\tilde{v}_1 \doteq \frac{iv_1 - 1}{i},\]

\[\tilde{Y}_t \doteq Y_s + \int_s^t \left( r_u - \frac{\tilde{\sigma}^2}{2} \right) du + \tilde{\sigma} \int_s^t dW_u\]
and split [A.6] into the product of components

\begin{align}
C_s(\tau) &= \frac{1}{P_s(s + \tau)} \exp \left( Y_s + \frac{\sigma \tilde{\sigma}}{2} \tau \right), \\
D_s(v_1, v_2, \tau) &= \mathbb{E}_s \left( \exp \left( i\tilde{v}_1 \left( Y_s + \int_s^{s + \tau} \left( r_u - \frac{\tilde{\sigma}^2}{2} \right) du + \tilde{\sigma} \int_s^{s + \tau} dW_u \right) + iv_2 r_{s+\tau} \right) \right) \\
&= \mathbb{E}_s \left( \exp \left( i\tilde{v}_1 \tilde{Y}_{s+\tau} + iv_2 r_{s+\tau} \right) \right), \\
F_s(v_1, \tau) &= \exp \left( -i\lambda \left( \phi_L (-i) - 1 \right) v_1 \tau \right) \mathbb{E}_s \left( \exp \left( iv_1 \int_\mathbb{R} l \left( N_{s+\tau} (dl) - N_{s} (dl) \right) \right) \right) \\
&= \exp \left( \lambda \left( -i \left( \phi_L (-i) - 1 \right) v_1 + \phi_L (v_1) - 1 \right) \tau \right),
\end{align}

where [A.9] is due to the Lévy-Khintchine formula applicable on the jump component of the jump diffusion process.

As far as [A.8] is concerned, we apply the characterization of regular affine Markov processes provided in Duffie et al. (2003) as illustrated in Kallsen (2006). To start with, the pair \( \left( r, \tilde{Y} \right) \) forms an affine process, in the sense of Kallsen (2006), equation 3.1). Hence, the joint \( \left( r, \tilde{Y} \right) \) distribution can be deduced uniquely as the solution to an affine martingale problem. Then, from Kallsen (2006), Theorem 3.2), we have, for the system of stochastic differential equations

\begin{align}
\frac{dr_t}{dt} &= -\kappa (r_t - \mu_r) \, dt + \sigma_r \, dW_{r,t} \\
\frac{d\tilde{Y}_t}{dt} &= \left( r_t - \frac{\tilde{\sigma}^2}{2} \right) \, dt + \tilde{\sigma} \, dW_t,
\end{align}

that

\begin{align}
D_s(v_1, v_2, \tau) &= \exp \left( E_0 (v_1, v_2, \tau) + E_1 (v_1, v_2, \tau) r_s + E_2 (v_1, v_2, \tau) Y_s \right),
\end{align}

where [A.9] is due to the Lévy-Khintchine formula applicable on the jump component of the jump diffusion process.
where $E_0$, $E_1$ and $E_2$ satisfy the subsequent system of generalized Riccati equations:

\[
\begin{align*}
\frac{\partial E_0(v_1, v_2, u)}{\partial u} &= \kappa \mu_r E_1(v_1, v_2, u) - \frac{\bar{\sigma}^2}{2} E_2(v_1, v_2, u) + \frac{1}{2} \left( \sigma_r^2 E_1^2(v_1, v_2, u) \right. \\
&\quad + 2 \bar{\sigma} \sigma_r \rho E_1(v_1, v_2, u) E_2(v_1, v_2, u) + \bar{\sigma}^2 E_2^2(v_1, v_2, u) \left), \\
\frac{\partial E_1(v_1, v_2, u)}{\partial u} &= -\kappa E_1(v_1, v_2, u) + E_2(v_1, v_2, u), \\
\frac{\partial E_2(v_1, v_2, u)}{\partial u} &= 0,
\end{align*}
\]

\[E_0(v_1, v_2, s) = 0, \quad E_1(v_1, v_2, s) = iv_2, \quad E_2(v_1, v_2, s) = i\bar{v}_1,\]

under technical conditions (see Kallsen (2006), Definition 3.1). Given the fact that the logfirm value model contains no mean-reverting term and the interest rate process is of Ornstein-Uhlenbeck type, we explicitly obtain from Kallsen ((2006), Corollary 3.5) that

\begin{align}
E_2(v_1, v_2, \tau) &= i\bar{v}_1, \quad (A.11) \\
E_1(v_1, v_2, \tau) &= iv_2 \exp (-\kappa \tau) + \frac{1 - \exp (-\kappa \tau)}{\kappa} i\bar{v}_1 \\
&= iv_2 \left( \cosh (\kappa \tau) - \sinh (\kappa \tau) \right) + i\bar{v}_1 \left( 1 - \cosh (\kappa \tau) + \sinh (\kappa \tau) \right), \quad (A.12) \\
E_0(v_1, v_2, \tau) &= \int_s^{s+\tau} \left( \kappa \mu_r E_1(v_1, v_2, u) - \frac{i\bar{\sigma}^2 \bar{v}_1}{2} + \frac{1}{2} \left( \sigma_r^2 E_1^2(v_1, v_2, u) \right. \right. \\
&\quad + 2i\bar{\sigma} \sigma_r \rho E_1(v_1, v_2, u) \bar{v}_1 - \bar{\sigma}^2 \bar{v}_1^2 \left) \right) \, du \\
&= \frac{i\bar{v}_1}{\kappa} \left( \kappa \mu_r + i\bar{\sigma} \sigma_r \rho \bar{v}_1 + \frac{i\sigma_r^2 \bar{v}_1}{2\kappa} \right) \tau + \frac{i\bar{\sigma}^2 \bar{v}_1}{2} (i\bar{v}_1 - 1) \tau \\
&\quad + \frac{i}{\kappa} \left( v_2 - \bar{v}_1 \right) \left( \sinh (\kappa \tau) - \cosh (\kappa \tau) \right) + \left( \kappa \mu_r + i\bar{\sigma} \sigma_r \rho \bar{v}_1 + \frac{i\sigma_r^2 \bar{v}_1}{\kappa} \right) \\
&\quad - \frac{\sigma_r^2}{4\kappa} \left( v_2 - \frac{\bar{v}_1}{\kappa} \right)^2 \left( \sinh (2\kappa \tau) - \cosh (2\kappa \tau) + 1 \right), \quad (A.13)
\end{align}
Summarizing, equations (A.7) and (A.9)-(A.13) lead to

\[ \mathbb{E}_s^* \exp (iv_1 Y_{s+\tau} + iv_2 r_{s+\tau}) = C_s(\tau) \exp \left( E_0(v_1, v_2, \tau) + E_1(v_1, v_2, \tau) r_s \right. \]
\[ \left. + E_2(v_1, v_2, \tau) Y_s \right) F(v_1, \tau). \]

It is worth to note that the derived joint characteristic function (A.14) has been Fourier-inverted and efficiently compared against the joint density function, which is available in closed form for the \( \mathbb{R}^2 \)-valued Gaussian process \((r, Y)\).

Furthermore, upon considering the transformation

\[ (Z, Z_r) = (Y_{s+\tau} - Y_s + \ln P_s(s + \tau), r_{s+\tau} - r_s \exp (-\kappa \tau)), \]

we obtain that

\[ \phi^*(v_1, v_2) = \mathbb{E}^* \exp (iv_1 Z + iv_2 Z_r) \]

is independent of \( Y_s \) and \( r_s \).

In fact,

\[ \phi^*(v_1, v_2) = \exp \left( iv_1 (-Y_s + \ln P_s(s + \tau) - iv_2 r_s \exp (-\kappa \tau)) \right) \]
\[ \times \mathbb{E}_s^* \exp (iv_1 Y_{s+\tau} + iv_2 r_{s+\tau})) \]
\[ = \exp \left( iv_1 (-Y_s + \ln P_s(s + \tau)) - iv_2 r_s \exp (-\kappa \tau) \right. \]
\[ - \ln P_s(s + \tau) + Y_s + \frac{\sigma^2}{2} \tau + E_0(v_1, v_2, \tau) \]
\[ + E_1(v_1, v_2, \tau) r_s + E_2(v_1, v_2, \tau) Y_s \right) F(v_1, \tau) \]
\[ = \exp \left( \frac{\sigma^2}{2} \frac{iv_1}{iv_1 - 1} \tau + (iv_1 - 1) A_s(s + \tau) + E_0(v_1, v_2, \tau) \right) F(v_1, \tau), \]

where \( A_s(s + \tau) \) is given by equation [5]. The characteristic function of the log-firm value follows directly from equation (A.15).
D Cumulants of the log-firm value under the risk-neutral probability measure

In virtue of the Lévy-Khintchine formula, the risk-neutral characteristic function $\varphi$ of the log-return of the firm value, $\ln \frac{V_t}{V_0}$, is (under the assumption of constant interest rates)

$$\varphi (v) = e^{\psi (v) t} ,$$

$$\hat{\psi} (v) = i (r - \psi (-i)) v + \psi (v) ,$$

where

$$\psi (v) = - \frac{\sigma^2 v^2}{2} + \lambda (\phi_L (v) - 1) .$$

The $n$th-order cumulant of the log-return process can then be calculated as

$$c_n \left( \ln \frac{V_t}{V_0} \right) = \frac{t}{i^n} \frac{\partial^n \hat{\psi} (v)}{\partial v^n} \bigg|_{v=0} .$$

Hence,

$$\mathbb{E} \left( \ln \frac{V_t}{V_0} \right) = (r - \psi (-i) + \lambda \mathbb{E} (L)) t ,$$

$$\mathbb{V} \text{ar} \left( \ln \frac{V_t}{V_0} \right) = (\sigma^2 + \lambda \mathbb{E} (L^2)) t ,$$

$$s \left( \ln \frac{V_t}{V_0} \right) = \frac{\lambda \mathbb{E} (L^3)}{\sqrt{\sigma^2 + \lambda \mathbb{E} (L^2)}} t ,$$

$$\kappa \left( \ln \frac{V_t}{V_0} \right) = \frac{\lambda \mathbb{E} (L^4)}{(\sigma^2 + \lambda \mathbb{E} (L^2))^2} t .$$

The formulae of the raw moments $\mathbb{E} (L^n)$, $n = 1, 2, 3, 4$, for the processes considered in this paper are summarized in Table 9.
Table 9: Raw moments of the jump sizes under the MJD and the DEJD models.

<table>
<thead>
<tr>
<th>Moment</th>
<th>MJD</th>
<th>DEJD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E}(L)$</td>
<td>$\mu_L$</td>
<td>$\frac{\mu}{1} - \frac{\mu}{\eta}$</td>
</tr>
<tr>
<td>$\mathbb{E}(L^2)$</td>
<td>$\mu_L^2 + \sigma_L^2$</td>
<td>$2 \left( \frac{\mu}{\eta_1} + \frac{\mu}{\eta_2} \right)$</td>
</tr>
<tr>
<td>$\mathbb{E}(L^3)$</td>
<td>$3\sigma_L^2\mu_L + \mu_L^3$</td>
<td>$6 \left( \frac{\mu}{\eta_1} - \frac{\mu}{\eta_2} \right)$</td>
</tr>
<tr>
<td>$\mathbb{E}(L^4)$</td>
<td>$3\sigma_L^4 + 6\sigma_L^2\mu_L^2 + \mu_L^4$</td>
<td>$24 \left( \frac{\mu}{\eta_1} + \frac{\mu}{\eta_2} \right)$</td>
</tr>
</tbody>
</table>

References


